CSE 599S Proof Complexity
Lecture 6 19 October 2020

Last time:

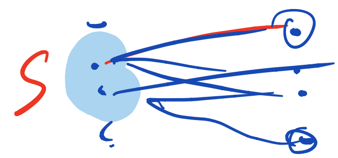
Thm (width to size lower bounds) small

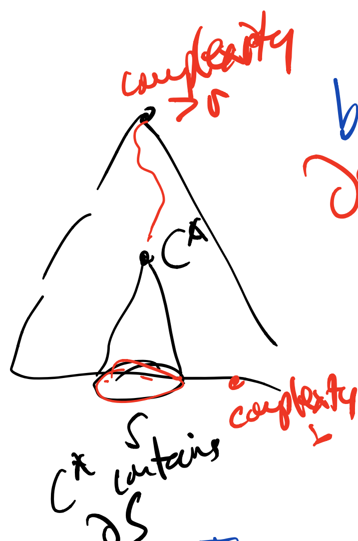
$$\text{Res}_{tree}(F) \geq 2^{\text{width}(F) - w(F)}$$

$$\text{Res}(F) \geq 2^{\frac{(\text{width}(F) - w(F))^2}{8n}}$$

for n variable F

Defn Graph G_F for CNF formula F





boundary of S: set of vars ∂S appearing in exactly one clause of S

G_F is an (r, c)-boundary expander iff $\forall S$ in C $|S| \leq r$ then $|\partial S| \geq c|S|$

Thm If G_F is an (r,c)-boundary expander then $\text{width}(F) \geq r \cdot c/2$

Note ⊛ is nearly optimal:
GT_n formula and replace
long clauses using extra vars
$(x_1 \vee x_2 \vee \dots \vee x_k)$

$\Rightarrow (x_1 \vee x_2 \vee y_2)(\neg y_2 \vee x_3 \vee y_3)$

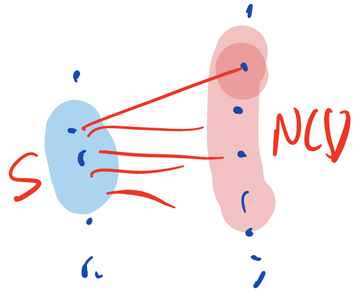
$\dots \quad (\neg y_{k-2} \vee x_{k-1} \vee x_k)$

extra var $y_2 \dots y_{k-2}$

get GT'_n all clauses size 3
poly size proof.

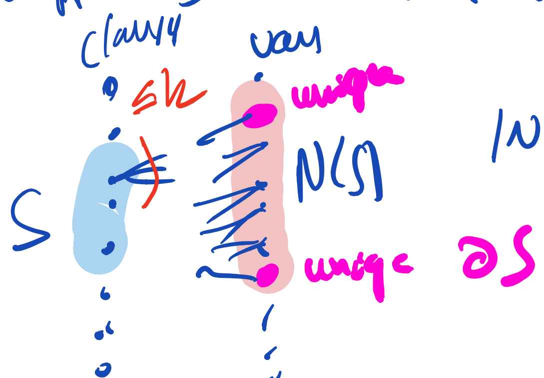
<u>Fact:</u> requires width n
in $n(n-1)$ vars.

Defn A bipartite $G=(L,R,E)$ is an (r,c)-<u>bipartite expander</u> iff every set $S \subseteq L$ with $|S| \le r$ has $\underbrace{|N(S)|}_{\text{neighborhood}} \ge (1+c)|S|$



Thm If F is a k-CNF formula and G_F is an (r,c)-bipartite expander then it is an (r,c')-boundary expander for $c' = 2(c+1) - k$

Proof Suppose S is a set of at most r clauses of F



$|N(S)| \ge (1+c)|S|$

$$k|S| \ge \underset{\text{touching } S}{\#\text{ of edges}} \ge |\partial S| + 2|N(S) - \partial S|$$
$$= 2|N(S)| - |\partial S|$$

$$|\partial S| \ge 2|N(S)| - k|S|$$
$$\ge 2(1+c)|S| - k|S|$$
$$= \underbrace{[2(1+c) - k]}_{c'}|S|$$

∎

All we need is to prove that
G_F is a (r,c)-bipartite expander
for $c = \frac{k}{2} - 1 + \delta$

$\Rightarrow c' = 2\delta$

$\Rightarrow \text{width}(F) \geq r\delta$

Fact in notes

If F is an (r,c)-bipartite expander then

$\text{width}(F) > rc/2$.

Use Hall's **Theorem**:

If bipartite graph has $|N(S)| \geq |S|$ for every subset S then it has a perfect matching

$\Leftarrow$ Formula is SAT

Hard examples

Random k-CNF formulas in n vars

- Choose $m = \Delta n$ clauses independently at random

defn $\mathcal{F}_n^{k,m}$ length exactly k.

uniform on all

$2^k \binom{n}{k}$ possible clauses

$\underset{\text{signs}}{2^k} \underset{\text{vars}}{\binom{n}{k}}$

Theorem Let $k \geq 2$ be an integer and

$\Delta > 2^k \ln 2$

$F \sim \mathcal{F}_n^{k,m}$ for $m = \Delta n$.

F is unsatisfiable with prob $1 - o_n(1)$

$k=3$

$\Delta \quad 5.28\ldots$

Proof Let $\varepsilon = \Delta - 2^k \ln 2$

Fix truth asst α

$\Pr[F(\alpha) = 1]$?

$\Pr[C(\alpha) = 1]$ for C a k-clause

$= 1 - \frac{1}{2^k}$

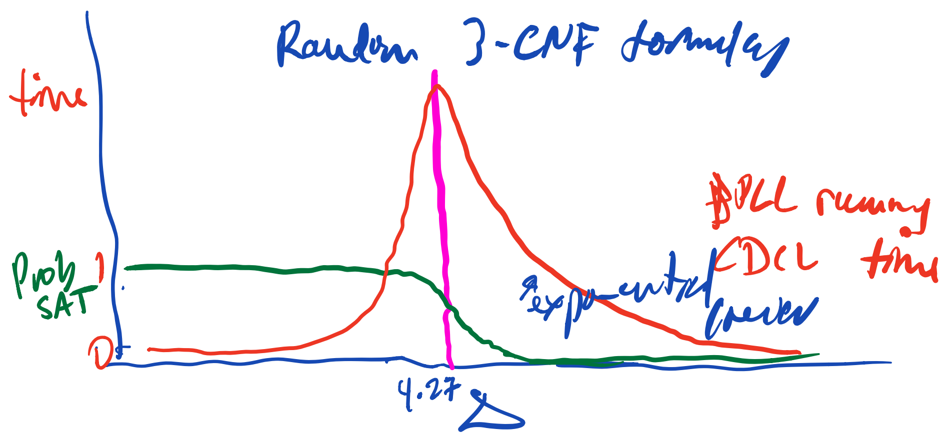
$$\Pr(F(\alpha)=1) = \prod_{C\in F} \Pr[C(\alpha)=1]$$
$$= \prod_{C\in F}\left(1-\frac{1}{2^k}\right)$$
$$= \left(1-\frac{1}{2^k}\right)^m$$

$$\Pr[F \text{ is SAT}] = \Pr[\cancel{\exists}\alpha.\ F(\alpha)=1]$$
$$\le \sum_{\alpha\in\{0,1\}^n} \Pr[F(\alpha)=1]$$
$$= 2^n\cdot\left(1-\frac{1}{2^k}\right)^m$$
$$\le 2^n e^{-m/2^k}$$
$$\le 2^n e^{-\Delta n/2^k}$$
$$= 2^n e^{-(\varepsilon+2^k\ln 2)n/2^k}$$
$$= 2^n e^{-\frac{\varepsilon n}{2^k}}\cdot e^{-(\ln 2)n}$$
$$= e^{-\varepsilon n/2^k} \xrightarrow[n]{} 0$$

$\boxtimes$

$$\boxed{1+x\le e^x}$$

$$e^{-\ln 2}=\frac{1}{2}$$



$CH \gg \frac{k}{2}$

Thm Random 3-CNF formulas are $(r, 3/4)$ bipartite expanders w.h.p. $1 - o_n(1)$
where r is linear in n
$\Rightarrow$ width $\Omega(n)$

Proof

S $(1+\frac{3}{4})s = \frac{7}{4}s$

S $T = q$

m n

$|S| = s$

Fix a set S of size s.
a set T of size $q = \frac{7s}{4}$

Prob all clauses of S contain variables from T

$\Pr[\text{all of } C\text{'s vars are in } T]$

$$= \frac{\binom{q}{k}}{\binom{n}{k}} \le \left(\frac{q}{n}\right)^k = p$$

define

What is prob all s clauses of S have vars only in T $\le p^s$

$$\Pr[\,|N(S)| \le q\,] \le \binom{n}{q} p^S$$

choice of T

$$= \binom{n}{q}\left(\frac{q}{n}\right)^{ks} \qquad \underline{k=3}$$

$$\le \left(\frac{ne}{q}\right)^q \left(\frac{q}{n}\right)^{ks}$$

$$= e^q \left(\frac{q}{n}\right)^{3s-q}$$

Fact: $k! \ge \left(\frac{k}{e}\right)^k$

so $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$

$\binom{n}{k} \le \frac{n^k}{k!}$

$$\left(3-\frac{7}{4}\right)=$$

$$= e^{\frac{7}{4}s} \left(\frac{7}{4}\cdot\frac{s}{n}\right)^{5s/4}$$

$$= a^s \left(\frac{s}{n}\right)^{5s/4}$$

for some constant a.

$$a = e^{7/4}\cdot\left(\frac{7}{4}\right)^{5/4}$$

Pr G_F is not an (r,c)-bipartite

expand...

$$\sum_{s=1} \binom{\Delta n}{s} \cdot a^s \left(\frac{s}{n}\right)^{5s/4}$$

choices of S

$$= \sum_{s=1}^{r} \left(\frac{e\Delta n}{s}\right)^s a^s \left(\frac{s}{n}\right)^{5s/4}$$

$$= \sum_{s=1}^{r} e^s \Delta^s a^s \left(\frac{s}{n}\right)^{s/4}$$

$$= \sum_{s=1}^{r} b^s \left(\frac{s}{n}\right)^{s/4}$$

$$= \sum_{s=1}^{r} \left(b\left(\frac{s}{n}\right)^{1/4}\right)^s$$

$$\le \Big(\underbrace{b\left(\frac{r}{n}\right)^{1/4}}_{\le \frac{1}{2}}\Big)^s$$

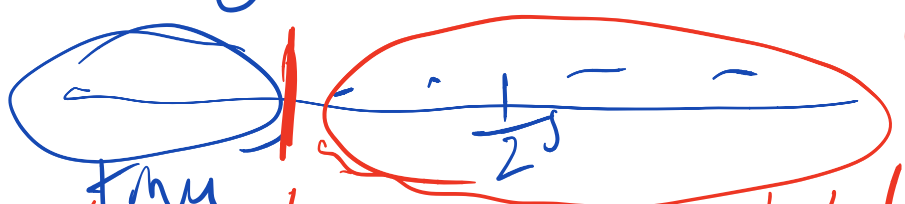
for $r \le \frac{n}{(2b)^4}$

for s small

$\downarrow$

0

$s \le \log n$



tiny $\to 0$

$t = \log n$

total $\Rightarrow \frac{2}{2^t} \to 0$

Thm For a random k CNF
is exponentially hard
for when # of clauses
Res
$\leq n^{\frac{k}{2}-\delta}$

$2^{\Omega(n^{\delta/2})}$

k=3 $n^{\frac{3}{2}-\delta}$

Res_{tree} $\leq n^{k-1-\delta}$ clauses

(k=3) $n^{2-\delta}$

known can prove unsat
(not by resolution) if
# of clauses is
$\Omega(n^{k/2})$
Eigenvalue methods

$k=3$

$cn \rightarrow n^{3/2-\varepsilon}$

hard for resolution

open for algs

$\sim n^{1.1}$ Feige

Feige's Conjecture NP-hard to find proofs for random formulas in cn clauses

Alan Sly $k \to \infty$

threshold: $2^k \ln 2 - \frac{1}{2}(1+\ln 2) - o(1)$

STOC 2015